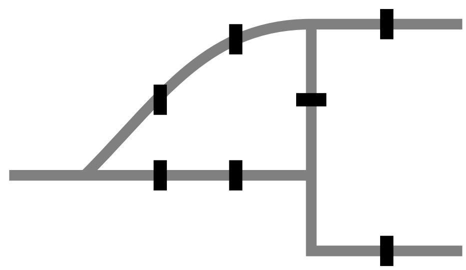
# Chapters 4.4 Eulerian Graph

Historical problem: Take a walk in Königsberg and traverse every bridge exactly once. Bridges are black, rivers are gray.

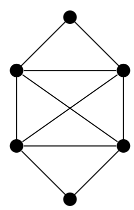


**1:** Is it possible to traverse every bridge exactly once?

**Solution:** No

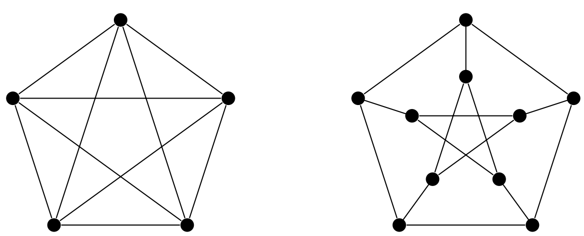
A **trail** is a walk without repeating edges. A **circuit** is a closed trail. A graph is **Eluerian** if it contains a circuit that contains all edges. Such circuit is called an **Eulerian circuit**.

**2:** Find Eulerian circuit in the following graph



**Solution:**

**3:** Decide if $K_5$ and the Petersen's graph are Eulerian.



**Solution:** $K_5$ is and Petersen's is not.

**4:** Show that if $G$ is Eulerian, then degree of every vertex is even.

**Solution:** Let $v$ be any vertex of $G$ and $C$ be the circuit. Traverse the circuit and notice that the sequence always looks like $eve'$, where $e$ and $e'$ are edges containing $v$. Since the circuit is closed, the edges incident to $v$ always come in pairs.

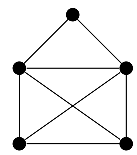
**Theorem 4.4.1** A nontrivial connected graph $G$ is Eulerian if and only if every vertex of $G$ has even degree.

**5:** Show that if a connected graph $G$ has every vertex of even degree, then $G$ is Eulerian. (Hint: Take longest circuit, induction on number of edges.)

**Solution:** Use induction on number of edges in $G$, base case is $C_k$. In induction step take the longest circuit $C$. If not all edges covered, there is a vertex $x$ in $C$ that is incident to some uncovered edge. Consider graph $G' = (V(G), E(G) \setminus E(C))$. Notice $G'$ has all vertices of even degree and less edges than $G$. Take a component of $G'$ that contains $x$. By induction, it has an Eulerian circuit $C'$. Notice $C'$ can be inserted in $C$, which contradicts maximality of $C$.

An **Eulerian trail** in a graph $G$ is a trail in $G$ containing all edges and does not start and end at the same vertex.

**6:** Find an Eulerian trail in the following graph



**Corollary** A connected graph $G$ contains an Eulerian trail if and only if exactly two vertices of $G$ have odd degree. Furthermore, each Eulerian trail of $G$ begins at one of these odd vertices and ends at the other.

**7:** Prove The Corollary using Theorem 4.4.1

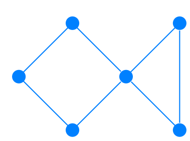
**Solution:** Let $a$ and $b$ be the two odd vertices. Create a graph $G'$ from $G$ by adding a new edge $e = ab$. This graph has an Eulerian circuit $C$. Observe that $C - e$ will result in Eulerian trail. (What if $ab$ is already edge in $G$?)

**8:** Does every Eulerian bipartite graph have an even number of edges? Explain.

**Solution:** Yes. Let the graph has partite sets $A$ and $B$. Partition the edges on the circuit to edges traversed from $A$ to $B$ and from $B$ to $A$. Since these two kinds alternate, there must be same number of them. This gives that the number of edges is even.

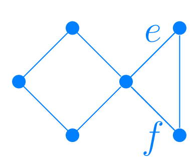
**9:** Does every Eulerian graph with an even number of vertices have an even number of edges? Explain.

**Solution:**



**10:** Prove or disprove the following statement: If $G$ is an Eulerian graph with edges $e$ and $f$ that share a common vertex $v$, then there is an Eulerian circuit which goes through the edge $e$ and then immediately after through $f$.

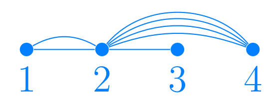
**Solution:**



A **multigraph** is $G = (V, E)$, where $V$ is the set of vertices and $E$ is a multiset of unordered pairs of vertices.

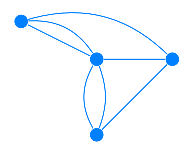
**11:** Draw the following multigraph $G = (\{1, 2, 3, 4\}, \{\{1, 2\}, \{2, 3\}, \{1, 2\}, \{2, 4\}, \{2, 4\}, \{2, 4\}, \{2, 4\}\})$

**Solution:**




**12:** Formulate Königsberg problem as a graph theory problem for Eulerian circuit. Can you do it both as a (simple) graph and a multigraph?

**Solution:** Normally, it would be a multigraph, but one can do some subdivisions of edges to make it easier. That means, put a new vertex in the middle of every edge.



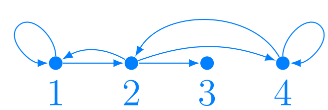
Notice that the trail in some sense has a direction. Hence Eulerian graph can be defined also for graph with "one-way" edges. directed (or oriented) graphs.

A **directed graph** is $D = (V, E)$, where $V$ is the set of vertices and $E \subseteq V \times V$. That means, edges are ordered pairs of vertices, not necessarily distinct.

In a directed graph, edge $(v, v)$ is called a **loop**. For a vertex $v$, its **out-degree** is $\deg^+(v) = |\{(v, u) \in E\}|$ and **in-degree** is $\deg^-(v) = |\{(u, v) \in E\}|$.

**13:** Depict the following directed graph $D = (\{1, 2, 3, 4\}, \{(1, 2), (2, 3), (2, 1), (2, 4), (4, 2), (4, 4), (1, 1)\})$. How to indicate the direction of an edge? For all four vertices, write that is their $\deg^+$ and $\deg^-$.

**Solution:**




A simple graph with orientation on edges is called an **oriented graph**. It has no loop and at most one of $(u, v), (v, u)$. In other words, you take a simple graph and orient the edges.

**14:** Show that a directed graph $D$ is Eulerian if and only if the graph is connected and at each vertex the in-degree equals the out-degree.

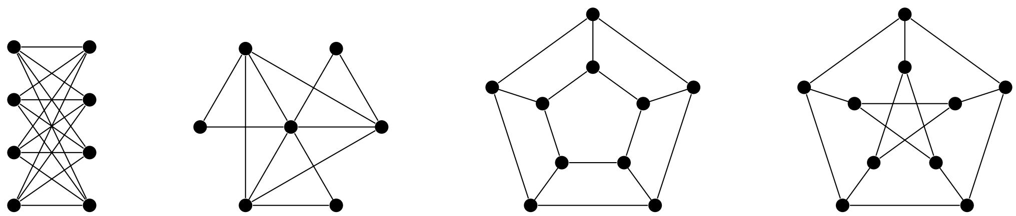
**Solution:** One can use exactly the same proof as for the undirected case.

**15:** Prove that if $P$ and $Q$ are paths of maximum length in a connected graph $G$, then $P$ and $Q$ have at least one vertex in common.

**Solution:** If not, take any path between any vertex of $P$ and $Q$ and it creates a longer path.

A graph $G$ on $n$ vertices is **Hamiltonian** if it contains a cycle of length $n$. The cycle is called **Hamiltonian cycle**. (Imagine you want to visit every vertex of a graph once. You don't care about edges.) A path containing all vertices is called **Hamiltonian path**.

**16:** Decide for the following graphs if they are Hamiltonian, have Hamiltonian path or nothing.



**Solution:** Both, Path, Both, Path

**17** *Open problem*: In a connected graph, pick any three paths of maximum length. Is there always a vertex that lies on all of them?